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SOME PROPERTIES AND APPLICATIONS OF CONVOLUTION ALGEBRAS

M.T. Garayev*

Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

Abstract. In the paper the classical double convolution product and the Duhamel product for functions in two variables is considered and some their properties and applications are studied. The special Banach space of the functions in the unit square is considered and it is proved that it is a Banach algebra with respect to the Duhamel product. Some its properties are studied and its maximal ideal space is described. Some applications of convolution and Duhamel products are given. The results given in this paper are extensions of our previous results.

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Corresponding author: M.T. Garayev, Department of Mathematics, College of Science, King Saud University, P.O.Box 2455, Riyadh 11451, Saudi Arabia, e-mail: *mgarayev@ksu.edu.sa*
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1 Introduction

Merryfield and Watson (1991) introduced the Duhamel product for the functions in two variables as follows :

$$(f \circledast g)(x,y) := \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y f(x-t, y-\tau)\, g(t,\tau) d\tau dt. \tag{1}$$

It is a natural extension of the Duhamel product on $\mathrm{Hol}(\mathbb{D})$ (see Wigley (1974)):

$$\begin{aligned} (f \circledast g)(z) &:= \frac{d}{dz} \int_0^z f(z-t)\, g(t) dt \\ &= \int_0^z f'(z-t)\, g(t) dt + f(0) g(z), \end{aligned} \tag{2}$$

where the integrals are taken over the segment joining the points 0 and z. In Karaev (2018), the author introduced the Duhamel algebra with respect to the Duhamel product (2) and studied its some properties and applications.

In the present paper we consider a special Banach space of functions in the unit square $J := [0,1] \times [0,1]$ and prove that it is a Banach algebra with respect to the Duhamel product (1) and study its some properties. Namely, we describe its maximal ideal space. We give some applications of convolution and Duhamel products (Section 3). Our results are extensions of the results of the paper Garayev et al. (2016). For the related results, see Garayev et al. (2016);

Saltan & Özel (2012, 2014) and for other applications of Duhamel product see Dimovski (1990); Ivanova & Melikhov (2017); Guediri et al. (2015); Gürdal (2009, 2015); Fage & Nagnibida (1987); Karaev (1987, 2011); Linchuk (2015); Tapdıgoglu (2012, 2013).

We will also consider the classical convolution product $*$ defined by

$$(f * g)(x,y) := \int_0^x \int_0^y f(x-t, y-\tau)\, g(t,\tau) d\tau dt \tag{3}$$

and characterize the $*$-generators of the algebra $C^{(n)}(J)$ with respect to this convolution product. Our results have certain interesting applications, namely we are able to exploit the underlying structure in order to establish an estimate for the solutions of some Volterra type integral equations in the terms of the kernel function (Section 3).

Recall that $C^{(n)}(J)$ is the space of two variables continuous functions in J with n $(n \geq 2)$ partial derivatives and the n^{th} continuous derivatives. We set

$$\|f\| = \sup_{y,x\in J} |f(x,y)|$$

for any continuous function f on J, and consider the norm

$$\|f\|_n := \max_{0\leq k\leq n} \left\| \frac{\partial^k}{\partial x^{k_1} \partial y^{k_2}} f(x,y) \right\| \tag{4}$$

for $f \in C^{(n)}(J)$, where $k = k_1 + k_2$.

2 A Banach algebra structure for $C^{(n)}(J)$ and its maximal ideal space

In the present section, we study the Banach algebra structure of the space $C^{(n)}(J)$ with respect to the Duhamel product (1). We set

$$C_{xy}^{(n)} := \left\{ f \in C^{(n)}(J) : f(x,y) = g(xy) \text{ for some single variable function } g \in C_{[0,1]}^{(n)} \right\}.$$

We prove that $\left(C_{xy}^{(n)}, \circledast\right)$ is a Banach algebra and describe its maximal ideal space. We start with the following lemmas.

Lemma 1. *The Banach space $C^{(n)}(J)$ is the commutative Banach algebra with respect to the Duhamel product* (1) *with the unity $f = \mathbf{1}$.*

Proof. Let $f, g \in C^{(n)}(J)$, $n \geq 2$. Then we have from (1) that

$$\begin{aligned} (f \circledast g)(x,y) &= \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} f(x-t, y-\tau)\, g(t,\tau)\, d\tau dt \\ &+ \int_0^x \frac{\partial}{\partial x} f(x-t, 0)\, g(t,y)\, dt \\ &+ \int_0^y \frac{\partial}{\partial y} f(0, y-\tau)\, g(x,\tau)\, d\tau + f(0,0)\, g(x,y). \end{aligned} \tag{5}$$

Using (4) and (5) it can be proved that (we omit the standards calculus)

$$\|f \circledast g\|_n \leq C_n \|f\|_n \|g\|_n, \tag{6}$$

where $C_n > 0$ is a constant depending only from n. By passing to equivalent norm, from inequality (6) we obtain that $\left(C^{(n)}(J), \circledast\right)$ is a Banach algebra. It is clear that $\mathbf{1} \circledast f = f$ for all $f \in C^{(n)}(J)$. Also, it is easy to see that $f \circledast g = g \circledast f$ for all $f, g \in C^{(n)}(J)$, i.e. $\left(C^{(n)}(J), \circledast\right)$ is a commutative Banach algebra with the unity $f = \mathbf{1}$. This proves the lemma. $\square$

Lemma 2. $\left(C_{xy}^{(n)}, \circledast\right)$ *is a commutative Banach algebra with the unity* $f = \mathbf{1}$.

Proof. Since $C_{xy}^{(n)}$ is a closed subspace of $C^{(n)}(J)$ and $(f \circledast g)(xy) \in C_{xy}^{(n)}$ for all $f, g \in C_{xy}^{(n)}$, it follows from Lemma 1 that $\left(C_{xy}^{(n)}, \circledast\right)$ is a commutative Banach algebra with the unity $f = \mathbf{1}$. Indeed, it follows from the formula (5) that

$$(f \circledast g)(x, y) = \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} f((x-t)(y-\tau)) g(t\tau) \, d\tau dt + f|_{xy=0} g(xy)$$

for all $f, g \in C_{xy}^{(n)}$ (because $\frac{\partial}{\partial x} f(0) = \frac{\partial}{\partial y} f(0) = 0$). So, the proof of Lemma 1 works. $\square$

The next lemma plays the central role in proving our main result.

Lemma 3. $f \in C_{xy}^{(n)}$ *is* $\circledast$*-invertible if and only if* $f|_{xy} \neq 0$.

Proof. In fact, we have for all $f, g \in C_{xy}^{(n)}$ that

$$(f \circledast g)(x, y) = \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} f((x-t)(y-\tau)) g(t\tau) \, d\tau dt + f|_{xy=0} g(xy). \tag{7}$$

If g is the $\circledast$-inverse of f we get

$$(f \circledast g)|_{xy=0} = f|_{xy=0} g|_{xy=0} = \mathbf{1},$$

where $f|_{xy=0} \neq 0$. Conversely, let $f|_{xy=0} \neq 0$. We set $\mathcal{D}_f(g) := f \circledast g$ for all $g \in C_{xy}^{(n)}$. We prove that $\mathcal{D}_f$ is an invertible operator on $C_{xy}^{(n)}$. To this aim, write f as $f = F + f|_{xy=0}$, where $F = f - f|_{xy=0} \in C_{xy}^{(n)}$ and $F|_{xy=0} = 0$. Thus, $\mathcal{D}_f = f|_{xy=0} I + \mathcal{D}_F$, where I is the identity operator on $C_{xy}^{(n)}$. Since $f|_{xy=0} \neq 0$, it suffices to prove that $\mathcal{D}_F$ is quasinilpotent, i.e., that $\sigma(\mathcal{D}_F) = \{0\}$.

For this purpose, we will show that

$$\lim_{k \to \infty} \left\| \mathcal{D}_F^k \right\|^{\frac{1}{k}} = 0.$$

In fact, we define for any $f \in C_{xy}^{(n)}$ the following usual convolution operator on $C_{xy}^{(n)}$:

$$(C_f g)(xy) := (f * g)(xy) = \int_0^x \int_0^y f((x-t)(y-\tau)) g(t\tau) \, d\tau dt. \tag{8}$$

Clearly, $\mathcal{D}_F = C_{\frac{\partial^2}{\partial x \partial y} F}$ (see formula (7)). Then, by considering that $F|_{xy=0} = 0$, we have

$$\begin{aligned}
(\mathcal{D}_F g)(xy) &= \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y F((x-t)(y-\tau)) g(t\tau) \, d\tau dt \\
&= \int_0^x \int_0^y \frac{\partial^2}{\partial x \partial y} F((x-t)(y-\tau)) g(t\tau) \, d\tau dt \\
&= C_{\frac{\partial^2}{\partial x \partial y} F} g(xy).
\end{aligned}$$

Thus, we get

$$
\begin{aligned}
&C^2_{\frac{\partial^2}{\partial x \partial y}F} g(xy) \\
&= \left(C_{\frac{\partial^2}{\partial x\partial y}F}\left(C_{\frac{\partial^2}{\partial x\partial y}F} g\right)\right)(xy) \\
&= \int_0^x \int_0^y \left(\frac{\partial^2}{\partial x\partial y}F\right)((x-t)(y-\tau)) \left(C_{\frac{\partial^2}{\partial x\partial y}F} g\right)(t\tau)\, d\tau dt \\
&= \int_0^x \int_0^y \left(\frac{\partial^2}{\partial x\partial y}F\right)((x-t)(y-\tau)) \left(\int_0^t \int_0^\tau \left(\frac{\partial^2}{\partial x\partial y}F\right)((t-u)(\tau-v))\right. \\
&= g(uv) dv du)\, d\tau dt.
\end{aligned}
$$

Consequently, we obtain

$$
\left|\left(C^2_{\frac{\partial^2}{\partial x\partial y}F} g\right)(xy)\right| \le \|F\|_n^2 \|g\|_n \frac{(xy)^2}{2!}.
$$

So, by induction we finally get

$$
\left|\left(C^k_{\frac{\partial^2}{\partial x\partial y}F} g\right)(xy)\right| \le \|F\|_n^k \|g\|_n \frac{(xy)^k}{k!}.
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C^2_{\frac{\partial^2}{\partial x\partial y}F} g\right)(xy) \\
= \quad & \int_0^x \int_0^y \left(\frac{\partial^{2+|\alpha|}}{\partial x^{1+\alpha_1}\partial y^{1+\alpha_2}}F\right)((x-t)(y-\tau)) \\
&\left(\int_0^t \int_0^\tau \left(\frac{\partial^2}{\partial x\partial y}F\right)((t-u)(\tau-v) g(uv)\, dv du\right) d\tau dt \\
&+ \left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}F\right)|_{xy=0} \int_0^x \int_0^y \left(\frac{\partial^2}{\partial x\partial y}F\right)((x-t)(y-\tau))\, g(t\tau) d\tau dt,
\end{aligned}
$$

where $|\alpha| = \alpha_1 + \alpha_2$ and $1 \le |\alpha| \le n$.

Thus, we obtain

$$
\begin{aligned}
\left|\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C^2_{\frac{\partial^2}{\partial x\partial y}F} g\right)\right)(xy)\right| \quad &\le \quad \|F\|_n^2 \|g\|_n \left(\frac{(xy)^2}{2!} + (xy)\right) \\
&\le \quad \|F\|_n^2 \|g\|_n \frac{(xy+1)^2}{2!}.
\end{aligned}
$$

Then, assume by induction that

$$
\left|\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C^k_{\frac{\partial^2}{\partial x\partial y}F} g\right)\right)(xy)\right| \le \|F\|_n^k \|g\|_n \frac{(xy+1)^k}{k!}.
$$

By differentation we have

$$
\begin{aligned}
&\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C^{k+1}_{\frac{\partial^2}{\partial x\partial y}F}g\right)\right)(xy)\\
= &\int_0^x\int_0^y\left(\frac{\partial^{2+|\alpha|}}{\partial x^{1+\alpha_1}\partial y^{1+\alpha_2}}F\right)((x-t)(y-\tau))\\
&\left(C^{k}_{\frac{\partial^2}{\partial x\partial y}F}g\right)(t\tau)d\tau dt+\left(\frac{\partial^{2+|\alpha|}}{\partial x^{1+\alpha_1}\partial y^{1+\alpha_2}}F\right)|_{xy=0}\left(C^{k}_{\frac{\partial^2}{\partial x\partial y}F}g\right)(xy).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\left|\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C^{k+1}_{\frac{\partial^2}{\partial x\partial y}F}g\right)(xy)\right| &\leq \|F\|_n^{k+1}\|g\|_n\left(\frac{(xy)^{k+1}}{(k+1)!}+\frac{(xy)^k}{k!}\right)\\
&\leq \|F\|_n^{k+1}\|g\|_n\frac{(xy+1)^{k+1}}{(k+1)!}.
\end{aligned}
$$

Now, from the equality

$$
\begin{aligned}
&\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C^{2}_{\frac{\partial^2}{\partial x\partial y}F}g\right)\right)(xy)\\
= &\int_0^x\int_0^y\left(\frac{\partial^{2+|\alpha|}}{\partial x^{1+\alpha_1}\partial y^{1+\alpha_2}}F\right)((x-t)(y-\tau))\left(C^{2}_{\frac{\partial^2}{\partial x\partial y}F}g\right)(t\tau)d\tau dt\\
&+\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}F\right)|_{xy=0}\left(C^{2}_{\frac{\partial^2}{\partial x\partial y}F}g\right)(xy),
\end{aligned}
$$

we infer that

$$
\begin{aligned}
&\left(\frac{\partial^{|\alpha|+1}}{\partial x^{\beta_1}\partial y^{\beta_2}}\left(C^{2}_{\frac{\partial^2}{\partial x\partial y}F}g\right)\right)(xy)\\
= &\int_0^x\int_0^y\left(\frac{\partial^{2+|\alpha|+1}}{\partial x^{\beta_1+1}\partial y^{\beta_2+1}}F\right)((x-t)(y-\tau))\left(C^{2}_{\frac{\partial^2}{\partial x\partial y}F}g\right)(t\tau)d\tau dt\\
&+\left(\frac{\partial^{2+|\alpha|}}{\partial x^{1+\alpha_1}\partial y^{1+\alpha_2}}F\right)|_{xy=0}\left(C_{\frac{\partial^2}{\partial x\partial y}F}g\right)\\
&+\left(\frac{\partial^{2}}{\partial x\partial y}\right)|_{xy=0}+\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\left(C_{\frac{\partial^2}{\partial x\partial y}F}g\right)\right)(xy),
\end{aligned}
$$

where $|\alpha|=\alpha_1+\alpha_2$ and $\beta_1+\beta_2=|\alpha|+1$, which leads to

$$
\begin{aligned}
\left(\frac{\partial^{|\alpha|+1}}{\partial x^{\beta_1}\partial y^{\beta_2}}\left(C^{2}_{\frac{\partial^2}{\partial x\partial y}F}g\right)\right)(xy) &\leq \|F\|_n^{2}\|g\|_n\left(\frac{(xy)^{2}}{2}+xy+\frac{(xy+1)^2}{2}\right)\\
&\leq \|F\|_n^{2}\|g\|_n\frac{(xy+2)^{2}}{2!}.
\end{aligned}
$$

Thus by induction we obtain

$$
\left|\left(\frac{\partial^{j}}{\partial x^{s}\partial y^{j-s}}\left(C^{k}_{\frac{\partial^2}{\partial x\partial y}F}g\right)\right)(xy)\right|\leq\|F\|_n^{k}\|g\|_n\frac{(xy+j)^2}{k!}
$$

for all $j \in \{2, ..., n\}$.

So, we have that

$$\left\| C^k_{\frac{\partial^2}{\partial x \partial y}F} g \right\|_n \leq \|F\|_n^k \|g\|_n \frac{(1+n)^k}{k!},$$

and thus

$$\left\| C^k_{\frac{\partial^2}{\partial x \partial y}F} \right\|^{\frac{1}{k}} \leq \|F\|_n \frac{n+1}{(k!)^{\frac{1}{k}}} \to 0 \ (k \to \infty).$$

This means that $C_{\frac{\partial^2}{\partial x \partial y}F}$ is quasinilpotent operator on $C_{xy}^{(n)}$, which implies that $\mathcal{D}_f$ is invertible in $C_{xy}^{(n)}$. The lemma is proved. □

Now we are ready to state the main result of the paper.

Theorem 1. $\left(C_{xy}^{(n)}, \circledast\right)$ *is a unital commutative Banach algebra with maximal ideal space* $\mathcal{M} = \{\varphi_0\}$, *where* $\varphi_0 : C_{xy}^{(n)} \to \mathbb{C}$ *and* $\varphi_0(f) = f|_{xy=0}$.

Proof. Let $\sigma(f)$ denote spectrum of the element f in the Banach algebra $\left(C_{xy}^{(n)}, \circledast\right)$. It follows from Lemma 3 that $\sigma(f) = \{f|_{xy=0}\}$ and by Gelfand's theory we see that $\mathcal{M} = \{\varphi_0\}$. In fact, the functions which vanish at the point $xy = 0$ form a maximal ideal. Any other proper ideal cannot have an element which does not vanish at $xy = 0$, hence there is only one maximal ideal. Consequently, the maximal ideal space $\mathcal{M}$ of $\left(C_{xy}^{(n)}, \circledast\right)$ consists of one homomorphism, namely evaluation at $xy = 0$, and the Gelfand transform is trivial. This proves the theorem. □

3 Applications

3.1 The $*$ generators of the radical algebra $\left(C_{xy}^{(n)}, *\right)$

Recall that for a Banach algebra $\mathcal{A}$ the radical $\mathcal{R}$ of $\mathcal{A}$ is a equal to the intersection of the kernel of all (strictly) irreducible representations of $\mathcal{A}$. If $\mathcal{R} = \{0\}$ then $\mathcal{A}$ is said to be semi-simple and if $\mathcal{R} = \mathcal{A}$, then $\mathcal{A}$ is called a radical algebra. Equivalently, $\mathcal{A}$ is a radical Banach algebra, if for every element $a \in \mathcal{A}$ the associated multiplication operator M_a, $M_a b := ab$ $(b \in \mathcal{A})$, is quasinilpotent on $\mathcal{A}$ (i.e., $\sigma(M_a) = \{0\}$).

It is classical that

$$\lim_{k\to\infty} \left\| f^{*k} \right\|_n^{\frac{1}{k}} = 0$$

and so $\left(C_{xy}^{(n)}, *\right)$ is a radical Banach algebra with respect to the convolution $*$ defined by means of formula (3) (see also (8)); here $f^{*k} := \overbrace{f*...*f}^{k}$ is the k^{th} iterated convolution of the function f in $\left(C_{xy}^{(n)}, *\right)$. For every $f \in C_{xy}^{(n)}$, we have that $(f * f)|_{xy=0} = 0$. Also

$$\begin{aligned} &(f * f * f)|_{xy=0} \\ = &\left(\int_0^x \int_0^y f((x-t)(y-\tau))\right) \int_0^t \int_0^\tau f((t-u)(\tau - v)) f(uv) dv du|_{xy=0} = 0. \end{aligned}$$

Thus, it can be easily shown that $f^{*k}|_{xy=0} = 0$, $k = 1, 2, ...$ and hence we see that a necessary condition for $f \in C_{xy}^{(n)}$ to generate $\left(C_{xy}^{(n)}, *\right)$ (i.e., to yield

$$\operatorname{span}\{f, f*f, f*f*f, ...\} = C_{xy}^{(n)}$$

is that $f|_{xy=0} \neq 0$). However, it is not yet known whether this condition is sufficient (for more fact about this type question, see Ginsberg and Newman Ginsberg & Newman (1970)). In this section, we discuss the above stated question in the Banach algebra $\left(C_{xy}^{(n)}, *\right)$, namely, we prove the following theorem, which reduces this question to the case of the subalgebra

$$C_{xy,0}^{(n)} := \left\{ f \in C_{xy}^{(n)} : f|_{xy=0} = 0 \right\}.$$

Theorem 2. *Let $f \in C_{xy}^{(n)}$ be a function such that $f|_{xy=0} \neq 0$. We set $F(x,y) := \int\limits_0^x \int\limits_0^y f(t\tau)d\tau dt$. Then f is $*$-generator of the Banach algebra $\left(C_{xy}^{(n)}, *\right)$ if and only if F is a $\circledast$-generator of the subalgebra $(C_{xy,0}^{(n)}, \circledast)$.*

Proof. In fact, since $F(x,y) = \int\limits_0^x \int\limits_0^y f(t\tau)d\tau dt$, we obtain for all $g \in C_{xy}^{(n)}$ that

$$\begin{aligned} \left(\mathcal{D}_F g\right)(xy) &= \frac{\partial^2}{\partial x \partial y} \int\limits_0^x \int\limits_0^y F\left((x-t)(y-\tau)\right) g(t\tau) d\tau dt \\ &= \int\limits_0^x \int\limits_0^y f\left((x-t)(y-\tau)\right) g(t\tau) d\tau dt. \end{aligned}$$

This shows that $\mathcal{D}_F = C_f$, where C_f is the couvolition operator defined above in formula (8). Hence, $F \circledast f = f * f$. Also, we have

$$(F \circledast F) \circledast f = \mathcal{D}_F^2 f = \mathcal{D}_F\left(\mathcal{D}_F f\right) = \mathcal{D}_F(C_f f) = C_f^2 f.$$

By induction we get $C_f^k f = \mathcal{D}_F^k f$ for all $k \geq 0$.

These equalities show that

$$\begin{aligned} & \operatorname{span}\left\{ f, f*f, f*f*f, ..., f\overbrace{*...*}^{m}f, ... \right\} \\ = & \operatorname{span}\left\{ f, F \circledast f, F \circledast F \circledast f, ..., F\underbrace{\circledast...\circledast}_{m-1}F \circledast f, ... \right\} \\ = & \operatorname{span}\left\{ \mathcal{D}_f\left(F^{\circledast k}\right) : k = 0,1,2,... \right\} \\ = & \operatorname{clos}\left\{ \mathcal{D}_f\left(\operatorname{span}\left\{ F^{\circledast k} : k = 0,1,2,... \right\}\right) \right\} \\ = & \operatorname{clos}\left\{ \mathcal{D}_f\left(\operatorname{span}\left\{ \mathbf{1}, F, F \circledast F, F \circledast F \circledast F, ... \right\}\right) \right\}. \end{aligned}$$

So, using the fact that

$$\begin{aligned} \operatorname{span}\left\{ \mathbf{1}, F, F \circledast F, F \circledast F \circledast F, ... \right\} &= \operatorname{span}\left\{ \lambda \mathbf{1} \colon \lambda \in \mathbb{C} \right\} \\ &\quad \oplus \operatorname{span}\left\{ F, F \circledast F, F \circledast F \circledast F, ... \right\}, \end{aligned}$$

where $\oplus$ stands for the direct sum of subspaces, we have that

$$\begin{aligned} \operatorname{span}\left\{ f, f*f, f*f*f, ... \right\} = \operatorname{clos}\{ \mathcal{D}_f (& \operatorname{span}\left\{ \lambda 1 : \lambda \in \mathbb{C} \right\} \\ & \oplus \operatorname{span}\left\{ F, F \circledast F, F \circledast F \circledast F, ... \right\}) \}. \end{aligned} \tag{9}$$

By considering that $f|_{xy=0} \neq 0$, due to Lemma 3 the Duhamel operator $\mathcal{D}_f$ is invertible on $C_{xy}^{(n)}$. On the other hand, by considering that

$$C_{xy}^{(n)} = \operatorname{span}\left\{ \lambda 1 : \lambda \in \mathbb{C} \right\} \oplus C_{xy,0}^{(n)}, \tag{10}$$

the assertion of the theorem now follows from the invertibility of the Duhamel operator $\mathcal{D}_f$ and representations (9) and (10). This completes the proof. □

3.2 An inequality for the solutions of the convolution equation

In this subsection, we prove an inequlity for the solutions of the convolution equation (i.e., the Volterra integral equation)

$$(C_K f)(xy) = \int_0^x \int_0^y K((x-t)(y-\tau)) f(t\tau) d\tau dt = g(xy) \tag{11}$$

in terms of the kernel function $K(xy) \in C_{xy}^{(n)}$. It is well known that equation (11) has a solution in the subspace $C_{xy}^{(n)}$ for any given function $g \in C_{xy}^{(n)}$. We set

$$\mathcal{G}_g := \left\{ u \in C_{xy}^{(n)} : u(x,y) \text{ is the solution of equation (11)} \right\}.$$

It is standard to prove that C_K is the Volterra operator, i.e., C_K is compact and $\sigma(C_K) = \{0\}$. Let $\sigma_p(C_K)$ denote the point spectrum of the operator C_K (i.e., the set of eigenvalues of C_K). Since C_K is compact, $\sigma_p(C_K) = \emptyset$. This implies that $g \notin \mathcal{G}$ for any nonzero $g \in C_{xy}^{(n)}$. Let $\mathcal{G}_g^1$ denote the unit sphere of the set $\mathcal{G}_g$, $\mathcal{G}_g^1 := \{u \in \mathcal{G}_g : \|u\|_n = 1\}$. The following problem naturally arizes :

To calculate the distance between g and G_g^1, denoted dist $\left(g, \mathcal{G}_g^1\right)$.

Our next result estimates dist $\left(g, \mathcal{G}_g^1\right)$ in terms of the kernel of function $K(xy)$.

Theorem 3. *We have:*

$$\inf \left\{ \text{dist}\left(g, \mathcal{G}_g^1\right) : g \in C_{xy}^{(n)} \setminus \{0\} \right\} \geq C_n^{-1} \left\| \left(-1 + \int_0^x \int_0^y K(t\tau) \, d\tau dt \right)^{-1\circledast} \right\|_n^{-1},$$

where $C_n > 0$ is the constant and the symbol $-1\circledast$ denotes the $\circledast-$inverse in the algebra $\left(C_{xy}^{(n)}, \circledast\right)$.

Proof. Denote $F(xy) := -1 + \int_0^x \int_0^y K(t\tau) d\tau dt$. Then the double convolution equation

$$\int_0^x \int_0^y K((x-t)(y-\tau)) u(t\tau) d\tau dt = g(xy)$$

can be rewritten as

$$\frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y F((x-t)(y-\tau)) u(t\tau) d\tau dt + u(xy) = g(xy),$$

or in brief as $F \circledast u = g - u$. Since $F|_{xy=0} = -1 \, (\neq 0)$, by Lemma 3, there exists a function $f \in C_{xy}^{(n)}$ such that $f \circledast F = \mathbf{1}$, which implies that

$$f \circledast F \circledast u = f \circledast (g-u),$$

that is $u = f \circledast (g - u)$. Hence by making use of Lemma 2, we obtain from (6) for any $u \in \mathcal{G}_g^1$ that

$$1 = \|u\|_n = \|f \circledast (g-u)\|_n \leq C_n \|f\|_n \|g-u\|_n,$$

which shows that

$$\|g-u\|_n \geq \frac{1}{C_n} \frac{1}{\|f\|_n}$$

for all $u \in \mathcal{G}_g^1$. Since $f = F^{-1\circledast}$, we infer that

$$\|g-u\|_n \geq \frac{1}{C_n}\frac{1}{\|F^{-1*}\|_n} = C^{-1}\left\|\left(-1+\int\limits_0^x\int\limits_0^y K(t\tau)d\tau dt\right)^{-1\circledast}\right\|_n^{-1}$$

for all $u \in \mathcal{G}_g^1$. Hence

$$\operatorname{dist}\left(g, \mathcal{G}_g^1\right) \geq C_n^{-1}\left\|\left(-1+\int\limits_0^x\int\limits_0^y K(t\tau)d\tau dt\right)^{-1\circledast}\right\|_n^{-1}. \tag{12}$$

By considering that $g \in C_{xy}^{(n)} \backslash \{0\}$ is arbitary, inequality (12) implies that

$$\inf\left\{\operatorname{dist}\left(g, \mathcal{G}_g^1\right) : g \in C_{xy}^{(n)} \backslash \{0\}\right\} \geq C_n^{-1}\left\|\left(-1+\int\limits_0^x\int\limits_0^y K(t\tau)d\tau dt\right)^{-1\circledast}\right\|^{-1}$$

which proves the theorem. □

References

Dimovski, I.H. (2012). *Convolutional Calculus* (Vol. 43). Springer Science & Business Media.

Ivanova, O., Melikhov, S. (2017). On operators commuting with a Pommiez type operator in weighted spaces of entire functions. *St. Petersburg Mathematical Journal*, *28*(2), 209-224.

Garayev, M.T., Guediri, H., Sadraoui, H. (2016). On some problems in the space $C^{(n)}[0,1]$. *U.P.B. Sci. Bull., Series A*, *78*(1), 147-156.

Ginsberg, J.I., Newman, D.J. (1970). Generators of certain radical algebras. *Journal of Approximation Theory, 3*(3), 229-235.

Guediri, H., Garayev, M.T., Sadraoui, H. (2015). The Bergman space as a Banach algebra. *New York J. Math*, 21, 339-350.

Gürdal, M. (2009). Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra. *Expositiones Mathematicae, 27*(2), 153-160.

Gürdal, M., Garayev, M.T., Saltan, S. (2015). Some concrete operators and their properties. *Turkish Journal of Mathematics*, *39*(6), 970-989.

Fage, M.K., Nagnibida, N.I. (1987). *The Problem of Equivalence of Ordinary Linear Differential Operators*, Novosibirsk, Russia, Nauka, Novosibirsk, Russia (in Russian).

Karaev, M.T. (2018). Duhamel algebras and applications. *Functional Analysis and Applications,* 52, 1-8.

Karaev, M., Gürdal, M., Saltan, S. (2011). Some applications of Banach algebra techniques. *Mathematische Nachrichten, 284*(13), 1678-1689.

Linchuk, Y. (2015). On derivation operators with respect to the Duhamel convolution in the space of analytic functions. *Mathematical Communications, 20*(1), 17-22.

Merryfield, K.G., Watson, S. (1991). A local algebra structure for H^p of the polydisc. *Colloq. Math.*, 62, 521-534.

Saltan, S., Özel, Y. (2012). On some applications of a special integrodifferential operators. *J. Funct. spaces and Appl.*, Article ID: 894527, 11 pages, doi:10.1155/2012/894527.

Saltan, S., Özel, Y. (2014). Maximal ideal space of some Banach algebras and related problems. B*anach. J. Math. Anal.*, *8*(2014), 16-29.

Tapdıgoglu, R. (2012). Invariant subspaces of Volterra integration operator: Axiomatical approach. *Bull. Sci. Mathematiques*, 136, 574-578.

Tapdıgoglu, R. (2013). On the description of invariant subspaces in the space $C^{(n)}[0,1]$. *Houston J. Math.*, *39*(2013), 169-176.

Wigley, N.M., (1974). The Duhamel product of analytic functions. D*uke Math. J.,* 41, 211-217.